

Excitation Spectrum at the Yang-Lee Edge Singularity of 2D Ising Model on the Strip

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Abstract

At the Yang-Lee edge singularity, finite-size scaling behavior is used to measure the low-lying excitation spectrum of the Ising quantum spin chain for free boundary conditions. The measured spectrum is used to identify the CFT that describes the Yang-Lee edge singularity of the 2D Ising model for free boundary conditions.

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In 1985, Cardy[1] identified the (A_4, A_1) minimal conformal field theory (CFT)[2, 3, 4] with the Yang-Lee edge singularity[5] of the 2-dimensional (2D) Ising model on the plane. Based on this identification, Cardy predicted properties of the Yang-Lee edge singularity.[1] Several of these CFT predictions have been confirmed numerically using the 2D Ising lattice model on the cylinder.[6, 7, 8]

The 2D Ising model can also be defined on surfaces having boundaries. On such surfaces, CFT on the $1/2$ -plane should describe critical properties,[9] e.g., at the Yang-Lee edge singularity. In general, different boundary conditions on the $1/2$ -plane will define different CFTs.[9] Each of these CFTs has a spectrum corresponding to one or more of the Verma modules of the corresponding CFT on the plane.[10]

The present article is organized as follows. First, it presents tools that were used to study the Yang-Lee edge singularity of the 2D Ising model. The tools include quantum spin chains and finite-size scaling. Second, it presents the excitation spectrum of candidate CFTs for the Yang-Lee edge singularity of the Ising model with free boundary conditions. Third, it presents finite-size measurements of the low-lying excitation spectrum at the Yang-Lee edge singularity of the Ising quantum spin chain with free boundary conditions. Fourth, it compares the measured spectrum to the spectra of the candidate CFTs.

To measure the excitation spectra, the extreme anisotropic limit is taken of the 2D Ising model. In this limit, each critical point of a 2D model becomes a critical point of a corresponding quantum spin chain.[11] In particular, each critical point of the 2D Ising model with free boundary conditions on the infinite strip will correspond to a critical point of the Ising quantum spin chain with free boundary conditions at its ends. The Yang-Lee edge singularities of these two models will have the same critical properties.

The Ising quantum spin chain for the 2D Ising model has a Hamiltonian given by:[12]

$$H_{Ising} = \sum_{n=1}^N \{-t\sigma_z(n)\sigma_z(n+1) - h\sigma_z(n) + \sigma_x(n)\}. \quad (1)$$

In eq. (1), N is the number of sites, $\sigma_x(n)$ and $\sigma_z(n)$ are 2×2 Pauli spin matrices at site " n ", " h " is an external magnetic field, and " t " is a ferromagnetic spin-spin coupling, i.e., $t > 0$.

At the Yang-Lee edge singularity, the external magnetic field, h , is purely imaginary, i.e., $h = iB$ for a real B . On a quantum spin chain of length, N , the phenomenological renormalization group (PRG) defines special values, $iB_{YL}(N)$, of the magnetic field. For a length of N , the special value $iB_{YL}(N)$ satisfies the PRG equation:[13, 6]

$$[N-1]m(iB_{YL}(N), N-1) = [N]m(iB_{YL}(N), N). \quad (2)$$

In eq. (2), $m(iB, N)$ is the energy gap, which is equal to $[E_1(iB, N) - E_0(iB, N)]$. The energies $E_0(iB, N)$ and $E_1(iB, N)$ are the energies of the respective ground state " 0 " and first excited state " 1 " for the Ising quantum spin chain of length N . As $N \rightarrow \infty$, the special values, $iB_{YL}(N)$, converge to the Yang-Lee edge singularity. The scaling behavior of physical quantities at solutions of the PRG equation provides the scaling behavior of same physical quantities near a corresponding critical point.[13, 6]

For a spin-spin coupling t of 0.1, Table [1] shows our measurements of the special values of the magnetic field that solve the PRG eq. (2). The special

N	$B_{YL}(N)$
6	0.64452828
7	0.64155254
8	0.63986508
9	0.63884984
10	0.63820860
11	0.63778681
12	0.63749969
∞	—

Table 1: PRG values of $B_{YL}(N)$ when $t = 0.1$.

values of the magnetic field of Table [1] correspond to the Yang-Lee edge singularity when free boundary conditions are imposed on the Ising quantum spin chain. When evaluated at the $B_{YL}(N)$'s, physical properties have finite-size scaling behaviors corresponding to the Yang-Lee edge singularity in the limit where $N \rightarrow \infty$.

In the same limit, CFT predicts that the leading scaling behavior of the actual excitation energies with the length, N , of a quantum spin chain will be:[14]

$$E_i(N) - E_0(N) = \xi 2\pi \Delta_i / N. \quad (3)$$

Here, Δ_i is the conformal dimension of the field "i", and ξ is non-universal constant that depends on the normalization of the Hamiltonian for the Ising quantum spin chain. In eq. (3), the excitation energies depend only on one conformal weight, because physical boundaries reduce the symmetry of the CFT to a single Virasoro algebra.

On the 1/2-plane, a CFT has a partition function that is a linear combination of the Virasoro characters appearing in the partition function of the same CFT on the plane.[9, 10] On the plane, the (A_4, A_1) minimal CFT describes the Yang-Lee edge singularity of the 2D Ising model. The partition function of the (A_4, A_1) minimal CFT is constructed from $(c = -22/5, \Delta = 0)$ and $(c = -22/5, \Delta = -1/5)$ Verma modules. Thus, the CFT of the Yang-Lee edge singularity on the infinite strip should have the states of the $(c = -22/5, \Delta = 0)$ Verma module and/or the states of the $(c = -22/5, \Delta = -1/5)$ Verma module. That is, there are three types of candidate CFT for the Yang-Lee edge singularity of the 2D Ising model on the infinite strip with free boundary conditions. The candidate CFTs have the states of the $(c = -22/5, \Delta = 0)$ Verma module, the states of the $(c = -22/5, \Delta = -1/5)$ Verma module, or the states of both these Verma modules. For each of these candidate CFT, Table [2] shows the low-lying excitation spectrum. In Table [2], the CFT predictions for the excitation energies are normalized so that the lowest excitation energy is two. This normalization removes both the dependency on the non-universal constant ξ and the dependence on the quantum spin chain's length, N .¹ For the types of CFT that combine both Verma modules,

¹For this choice of normalization, the measured normalized excitation energies will also

Strip Boundary Conditions						
CFT of $(-22/5, 0)$ Verma module						
Excitation Energy	2	3	4	5	6	7
Degeneracy	1	1	1	1	2	2
CFT of $(-22/5, -1/5)$ Verma module						
Excitation Energy	2	4	6	8	10	12
Degeneracy	1	1	1	2	2	3
CFT of $(-22/5, 0)$ and $(-22/5, -1/5)$ modules						
Excitation Energy	2	10	20	22	30	32
Degeneracy						
(A_4, A_1) CFT on ∞ -long cylinder						
Excitation Energy	2	5	10	12	15	
Degeneracy	1	2	3	2	4	

Table 2: Low-Lying Excitation spectra of candidate CFTs

Table [2] does not list state degeneracies, because these degeneracies will depend on the number of copies of each of the two Verma modules in the CFT. Table [2] also lists the low-lying excitation spectrum of the (A_4, A_1) minimal CFT, which describes the Yang-Lee edge singularity on the infinitely long cylinder. From Table [2], one also sees that the low-lying excitation spectra very substantially distinguish the candidate CFTs for the Yang-Lee edge from each other.

N	6	7	8	9	10	11	12	∞
1st excitation	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
A	3.10970	3.08865	3.07468	3.06481	3.05748	3.05183	3.04732	3.02
B	4.01911	4.01608	4.01552	4.01606	4.01700	4.01801	4.01895	4.02
C	4.72083	4.77954	4.82010	4.85112	4.87569	4.89555	4.91182	5.05
D1	—	5.35658	5.47758	5.56177	5.62651	5.67802	5.71994	6.01
D2	6.15534	6.12718	6.11081	6.09947	6.09136	6.08516	6.08017	6.06
E1	—	—	5.95864	6.13662	6.26129	6.35878	6.43745	6.90
E2	7.03192	7.02799	7.02804	7.03040	7.03483	7.03688	7.03914	7.03

Table 3: Measured Low-lying excitation energies as a function of length, N , of the Ising quantum spin chain.

For special magnetic field values and spin-spin coupling of Table [1], the low-lying excitation spectrum measured for Ising quantum spin chains of lengths between 6 and 12 sites is given in Table [3] and plotted in Figure (1). The measured excitation energies have been normalized by division by $1/2$ times the lowest measured excitation energy.² This normalization enables a direct comparison between the measured excitation spectra of Table [3] with the spectra

turn out to be actual conformal dimensions for this specific model.

²Figure (1) does not show the lowest excitation energy, which is exactly two in this normalization.

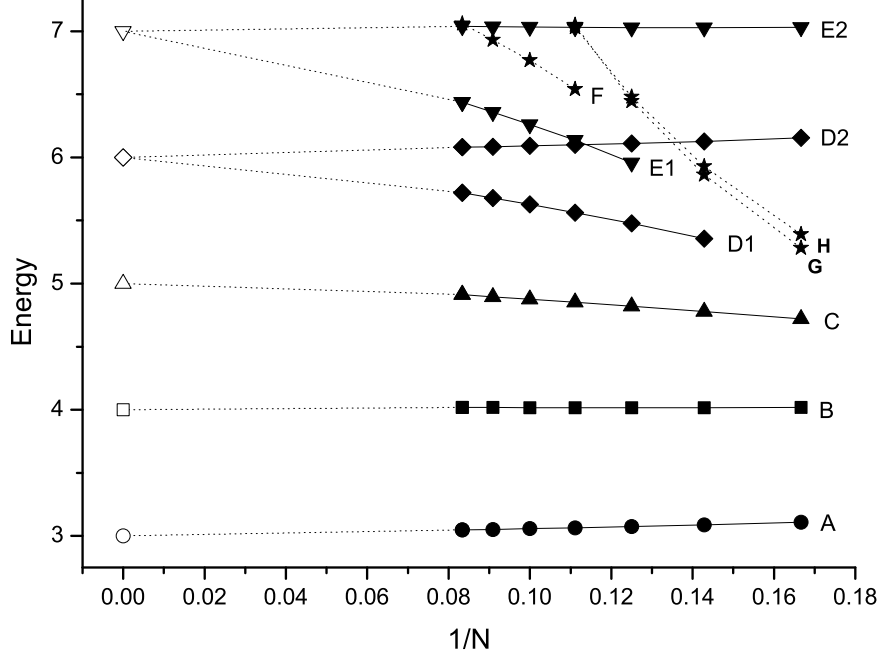


Figure 1: Measured Low-Lying Normalized Excitation Energies for N between 6 and 12.

of the candidate CFTs as given in Table [2].

To compare the measured spectra of Fig.(1) to the predicted spectra of Table [2], it is necessary to determine the limit of the normalized excitation energies of the states as the quantum spin chain's length, N , becomes large. To find these limit values, one must identify corresponding states in Ising quantum spin chains of different length, N . In Fig.(1), our conclusions about these correspondences are indicated by lines between measured excitation energies for different lengths, N . The correspondences were found by assuming a smooth scaling behavior and by using the fact that level crossings should disappear as N grows.

In Fig.(1), it is easy to identify sequences A, B, C, D1, D2, E1, E2, F, G, and H^3 of normalized excitation energies. As expected, crossings between different ones of these levels disappear as N increases. The normalized excitation energies of the lowest states, which are labeled by A, B, C, D1, D2, E1, and E2, smoothly scale toward values of about 3 to 7 as $N \rightarrow \infty$. The normalized excitation energies of the states labeled by F, G, and H smoothly scale towards

³No state has been double counted in identifying the sequences A - H.

higher values as $N \rightarrow \infty$. For the states F, G, and H, the evaluation of the limit of these higher energy excitations as $N \rightarrow \infty$ is outside of our measurements on Ising quantum spin chains of lengths between 6 and 12.

The last column of Table [3] also shows normalized excitation energies obtained by extrapolating the measured values to the limit where $N \rightarrow \infty$. Each such extrapolation was made by fitting normalized measured excitation energies of a corresponding state, i.e., $E_i(N) - E_0(N)$, to an equation of the following form:

$$E_i(N) - E_0(N) = [E_i(\infty) - E_0(\infty)] + A_i N^{p_i} \text{ with } p_i < 0. \quad (4)$$

In eq. (4), $[E_i(\infty) - E_0(\infty)]$ is the limit of the i -th excitation energy as $N \rightarrow \infty$, and $A_i N^{p_i}$ is a correction term whose form is motivated by finite-size scaling considerations.

From the above-described extrapolations, we find that the low-lying excitation spectrum for the Ising quantum spin chain at the Yang-Lee edge singularity has the following sequence of normalized excitation energies (degeneracies): 2(1), 3(1), 4(1), 5(1), 6(2), and 7(2). By comparing these results with the CFT predictions of Table [2], it is readily seen that the measured low-lying excitation spectrum is the same as the low-lying excitation spectrum of the minimal CFT that is based on the $(c, \Delta) = (-22/5, 0)$ Verma module.

In conclusion, the measured low-lying excitation spectrum of the Ising quantum spin chain with free boundary conditions at the Yang-Lee edge singularity is in excellent agreement with the low-lying spectrum of the $(c, \Delta) = (-22/5, 0)$ Verma module. Thus, the $(c, \Delta) = (-22/5, 0)$ Verma module defines the CFT for the Yang-Lee edge singularity of the 2D Ising model on the infinite strip with free boundary conditions.

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